

4.1 – Real Vector Spaces

Definition: (Vector space)

Let V be an arbitrary nonempty set of objects for which two operations are defined: addition and multiplication by numbers called **scalars**. By **addition** we mean a rule for associating with each pair of objects \mathbf{u} and \mathbf{v} in V an object $\mathbf{u} + \mathbf{v}$, called the **sum** of \mathbf{u} and \mathbf{v} ; by **scalar multiplication** we mean a rule for associating with each scalar k and each object \mathbf{u} in V an object $k\mathbf{u}$, called the **scalar multiple** of \mathbf{u} by k . If the following axioms are satisfied by all objects \mathbf{u} , \mathbf{v} , and \mathbf{w} in V and all scalars k and m , then we call V a **vector space** and we call the objects in V **vectors**.

1. If \mathbf{u} and \mathbf{v} are objects in V , then $\mathbf{u} + \mathbf{v}$ is in V .
 2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
 3. $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$
 4. There exists an object in V , called the **zero vector**, that is denoted by $\mathbf{0}$ and has the property that $\mathbf{0} + \mathbf{u} = \mathbf{u} + \mathbf{0} = \mathbf{u}$ for all \mathbf{u} in V .
 5. For each \mathbf{u} in V , there is an object $-\mathbf{u}$ in V , called a **negative** of \mathbf{u} such that $\mathbf{u} + (-\mathbf{u}) = (-\mathbf{u}) + \mathbf{u} = \mathbf{0}$.
 6. If k is any scalar and \mathbf{u} is an object in V , then $k\mathbf{u}$ is in V .
 7. $k(\mathbf{u} + \mathbf{v}) = k\mathbf{u} + k\mathbf{v}$
 8. $(k + m)\mathbf{u} = k\mathbf{u} + m\mathbf{u}$
 9. $k(m\mathbf{u}) = (km)(\mathbf{u})$
 10. $1\mathbf{u} = \mathbf{u}$
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#6 Determine whether the set equipped with the given operations is a vector space. For those that are not vector spaces, Identify the vector space axioms that fail.

The set of all n -tuples of real numbers that have the form (x, x, \dots, x) with the standard operations on R^n .

Examples of Vector Spaces

1. The simplest vector space is $\{0\}$.
2. R^n with the usual operations of addition and scalar multiplication of n -tuples
3. R^∞ , the set of infinite sequences of numbers with the usual operations of addition and scalar multiplication performed componentwise
4. The set of $m \times n$ matrices with matrix addition and scalar multiplication, denoted M_{mn}
5. The set of real-valued functions $f(x)$ that are defined for all $x \in R$, denoted $F(-\infty, \infty)$ with addition and scalar multiplication
 $(\mathbf{f} + \mathbf{g})(x) = f(x) + g(x)$ and $(k\mathbf{f})(x) = kf(x)$
6. The set of polynomials of degree $\leq n$, denoted P_n

